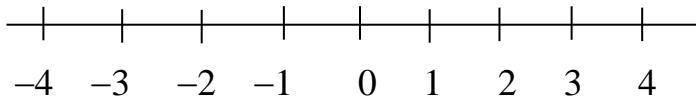


2. THE x - y PLANE

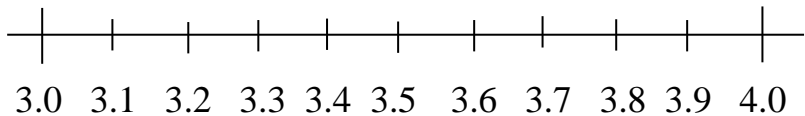
§2.1. The Real Line

When we plot quantities on a graph we can plot not only integer values like 1, 2 and -3 but also fractions, like $3\frac{1}{2}$ or $-4\frac{3}{4}$. In fact we can, in principle, plot any *real number*. Roughly speaking **real numbers** are positive, negative or zero numbers that can be represented by (possibly infinite) decimal expansions like 3.14159... If you can't think of any other sort of number that's O.K. There *are* other numbers, called **complex numbers**, but probably you haven't come across these yet.

Real numbers correspond to points on an infinite line. We choose one point, called the **origin**, to represent 0. Then we step out to the right in equal steps and call these points 1, 2, 3, ... Going left from the origin, using the same size steps, we mark off -1 , -2 , -3 , ...



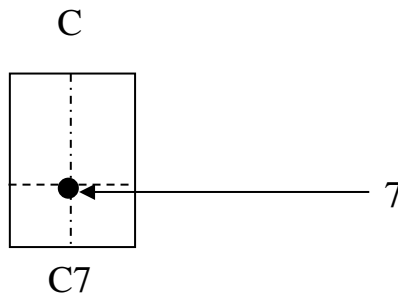
We can represent fractions by dividing these intervals into equal parts. In particular if we divide the interval from 3 to 4 into 10 equal parts we can plot subdivisions:



By dividing each of these tiny intervals again into tenths we can plot points corresponding to numbers like 3.14. By dividing and subdividing we can, in principle, find points that represent any real number.

§2.2. Two Axes Are Better Than One

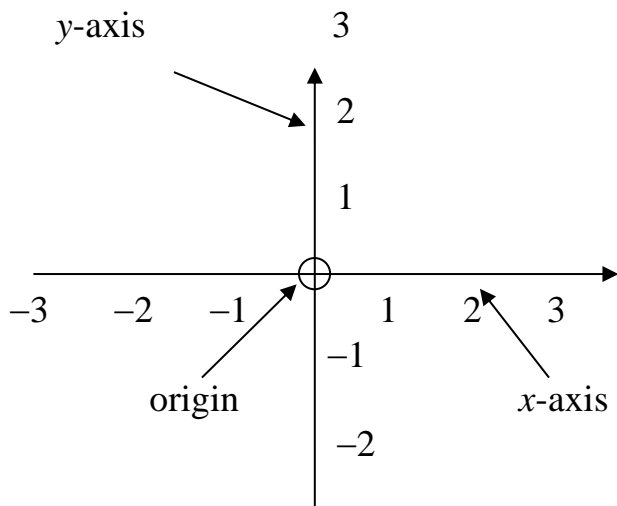
In the 17th century the French philosopher and mathematician, René Descartes [1596–1650], discovered that geometry could be done algebraically by introducing **coordinates**. We take it for granted in reading maps that we represent points on a map by a pair of letters or numbers where one gives the position in a left-right sense and the other gives the position in an up-down sense. So to find C7 we look where the C vertical and 7 horizontal intersect.



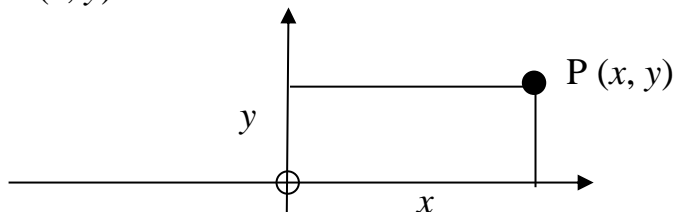
We navigate around a computer spreadsheet in a similar way. But in the 17th century it was quite a novel idea.

Because we want arbitrary precision we use two numbers, instead of a letter and a number. We take two copies of the number line, one horizontal and one vertical, and measure every point in the plane against them. The

horizontal axis, called the **x-axis**, has the numbers increasing from left to right. The vertical axis, called the **y-axis**, has numbers increasing from bottom to top. The whole plane is called the **x-y plane** and the point where the axes cut is called the **origin**.

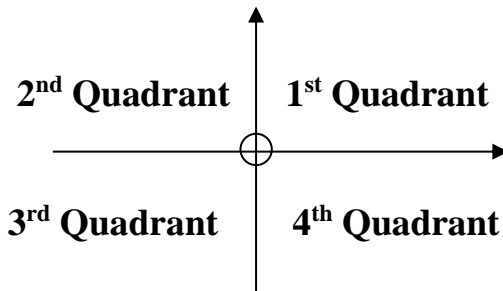


The two numbers that describe the position of a point P are the **x-coordinate** (measuring the horizontal position of P) and the **y-coordinate** (measuring the vertical position). We write the coordinates as an ordered pair (x, y) with the horizontal coordinate first.

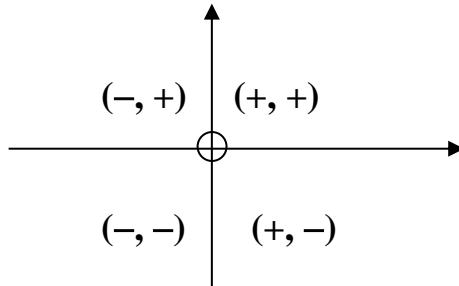


Sometimes it's convenient to use different scales on the axes, but it must be remembered that this will make the graph appear more steep or less steep than it should be. Here we'll generally assume that we're using the same scale on both axes.

The two axes divide the x - y plane into four **quadrants**. These are numbered, starting with the top right quadrant and moving around anticlockwise.



The points on the two coordinate axes could be considered to be in more than one quadrant but it's more usual to exclude them from the quadrants altogether. If we do this then we can say that the points in the 1st quadrant have positive x and y coordinates. Those in the 2nd quadrant have a negative x coordinate but a positive y coordinate, and so on. We summarise this as follows:



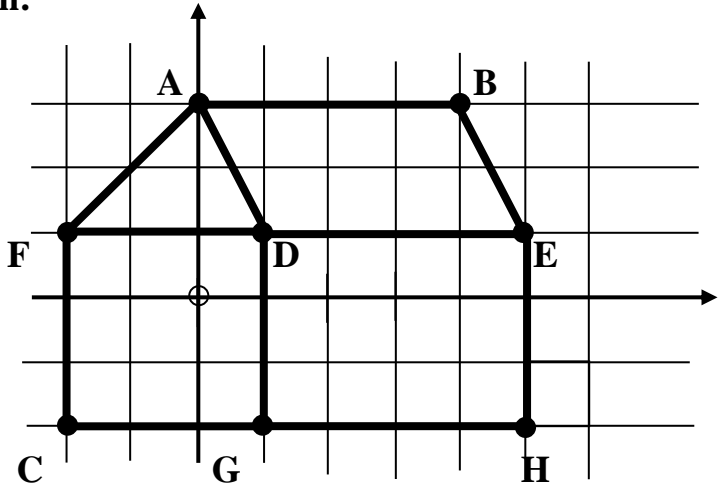
Example 1: Plot the following points on the x-y plane.

$A = (0, 3)$, $B = (4, 3)$, $C = (-2, -2)$, $D = (1, 1)$,

$E = (5, 1)$, $F = (-2, 1)$, $G = (1, -2)$, $H = (5, -2)$.

Now join AB , BE , DE , AD , AF , CF , DG , EH , CG , FD , GH by straight lines.

Solution:



§2.3. Functions and Their Graphs

Some graphs appear much smoother than others. These are usually graphs of functions that can be expressed by simple formulae. A **function** (here we limit ourselves to describing functions of a real variable) is a rule that associates with every real number x exactly one real number y . In principle the rule could be so complicated that it would take several pages to describe but here we only consider functions where the rule can be expressed by a single formula. The formula $y = x^2$ is a very compact way of expressing the rule that we take a real

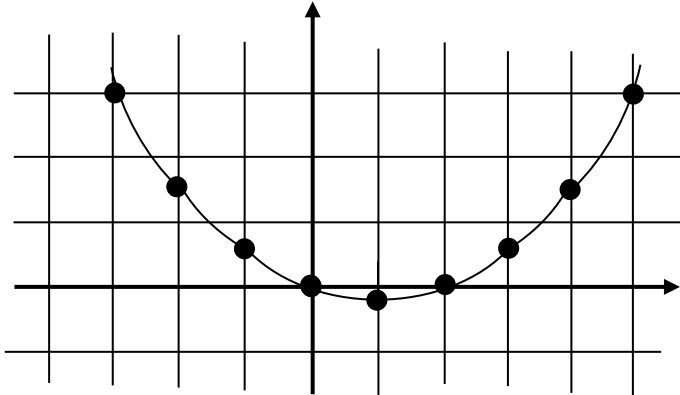
number x and multiply it by itself, the resulting square is y . So if $x = 3$ then $y = 9$.

It's not always the case that the relationship between two variables can be expressed by a simple formula. If there was a simple formula that gave the value of certain shares in terms of the number of months since it was listed on the Stock Exchange we'd all be millionaires. But in physics many phenomena behave according to reasonably simple formulae. Even in econometrics, while it might not be possible to describe particular events, it's possible to set up useful economic models that describe the whole economy in terms of systems of formulae. By programming these on a computer, economists can experiment with a mathematical model of the economy in a way that wouldn't be possible for the real economy. Of course we must point out that an economic model consists of many equations connecting a very large number of variables while we're only going to consider one equation connecting two variables. But we have to start somewhere! Every function, where y is given as a formula in x , can be represented as a graph, in the following way:

- (1) Substitute some values of x and come up with the corresponding values of y .
- (2) Set these out in a table of values.
- (3) Plot the corresponding points on the x - y plane.
- (4) Join the points by a smooth curve.

Example 2: $y = \frac{x^2 - 2x}{5}$

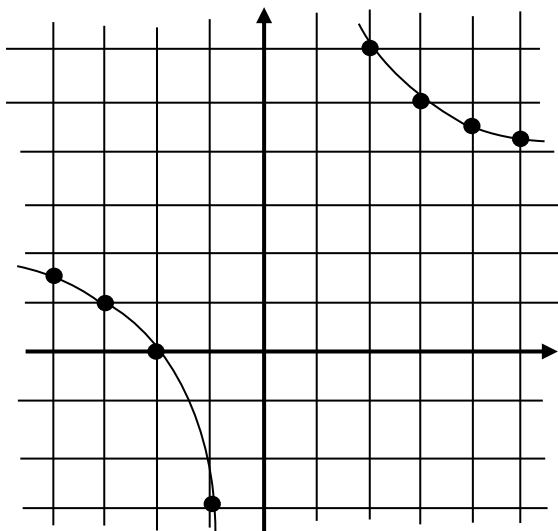
x	-3	-2	-1	0	1	2	3	4	5
y	3	1.6	0.6	0	-0.2	0	0.6	1.6	3



Example 3: $y = \frac{6}{x} + 3$

x	-4	-3	-2	-1	0	1	2	3	4	5
y	1.5	1	0	-3	×	9	6	5	4.5	4.2

Notice that there is no y value corresponding to $x = 0$.



§2.4 Straight Lines

The equation $y = 2x - 1$ represents a straight line. We'll see shortly that the general equation for a straight line is $y = mx + b$, where m and b are constants. But first let's consider a *segment*, that is, the part of a straight line that joins two points (x_1, y_1) and (x_2, y_2) .

Subscripts

Now we could have used a, b, c and d as the coordinates rather than x_1, y_1, x_2 and y_2 . But the subscript notation helps to remind us which are the x -coordinates and which are the y -coordinates, as well as which coordinates belong to the first point and which belong to the second. The important thing to remember is that these are double-barrelled names, just like people who have a family name and a given name. Here the x and the y don't

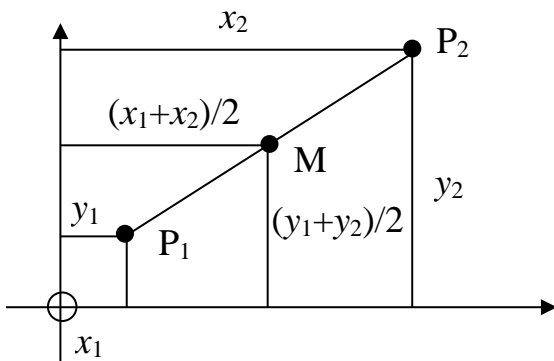
stand for anything by themselves and if we ever have $\frac{x_2}{y_2}$ we certainly can't cancel the 2's to get $\frac{x}{y}$. The little subscripts 1 and 2 must never be separated from their x or y .

Midpoint

The **midpoint**, M , of the segment from $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$ is clearly $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

<p>MIDPOINT</p> $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
--

Remember this as the average of the x -coordinates and the average of the y -coordinates.



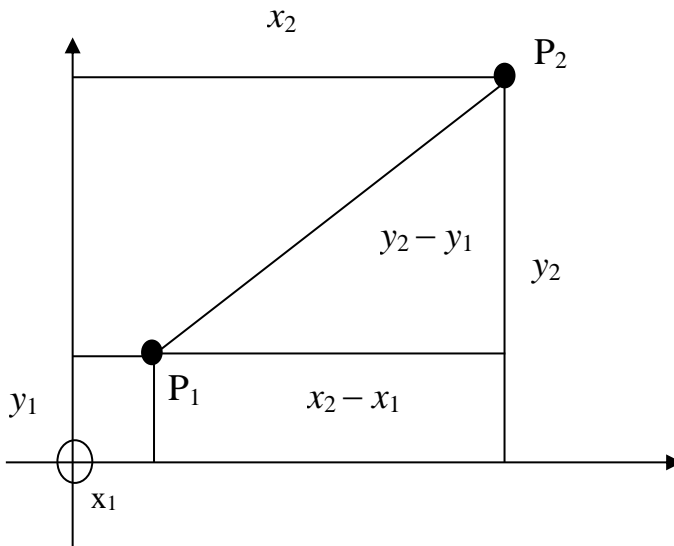
Example 4: The midpoint of the line segment joining (1, 6) to (9, 4) is $\left(\frac{1+9}{2}, \frac{6+4}{2}\right) = (5, 5)$.

Slope

The **slope** of the interval P_1P_2 is $\frac{\text{Rise}}{\text{Run}} = \frac{y_2 - y_1}{x_2 - x_1}$.

SLOPE
$\frac{y_2 - y_1}{x_2 - x_1}$

[If $x_1 = x_2$, the line is vertical and we say that the line has infinite slope.]



Example 5: The slope of the line segment joining (3, 5) to (7, 8) is $\frac{8-5}{7-3} = \frac{3}{4}$.

Distance

The **distance** between the points P_1 and P_2 is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

DISTANCE $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

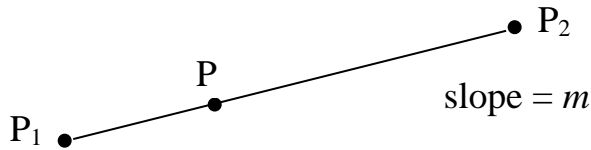
Take the horizontal difference and the vertical difference, square each one and add, and then take the square root of this sum. The reason for this is Pythagoras' Theorem (the square on the hypotenuse is the sum of the squares on the other two sides).

So $(P_1P_2)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.

Example 6: The distance between the points (3, 5) and (8, 4) is $\sqrt{(8-3)^2 + (4-5)^2} = \sqrt{25+1} = \sqrt{26}$.

One Point Plus Slope Equation of a Line

We're now in a position to work out the equation of a straight line. If we know the slope of the line and one point on the line we can find the equation as follows. Suppose the line passes through $P_1(x_1, y_1)$ and has slope m . If $P(x, y)$ is a typical point on the line then the slope of the line interval P_1P is equal to m .



So $\frac{y - y_1}{x - x_1} = m$.

We can multiply both sides by $x - x_1$ to get:

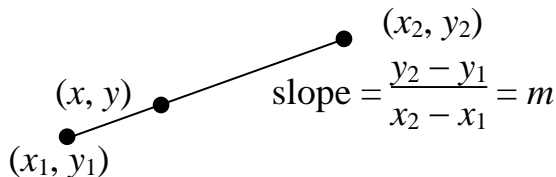
POINT + SLOPE EQUATION OF A LINE
 $y - y_1 = m(x - x_1)$

This is just one of many forms for the equation of a straight line but it's the most useful for our purposes.

Example 7: The equation of the line through (1, 5) with slope 3 is $y - 5 = 3(x - 1)$, ie $y = 3x + 2$.

Two Point Equation of a Line

Sometimes we don't know the slope but we're given two points on the line. Then all we do is to work out the slope and substitute into the $y - y_1 = m(x - x_1)$ formula.



The equation is thus $y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$.

But rather than having an extra formula to learn it's better to work out the slope first.

Example 9: The slope of the line through the points

$$(3, -1) \text{ and } (5, 2) \text{ is } \frac{2 - (-1)}{5 - 3} = \frac{3}{2}.$$

So the equation of the line through these points is:

$$y - 2 = \frac{3}{2}(x - 5), \text{ ie } y = \frac{3}{2}x - \frac{11}{2}$$

which can be written as $2y = 3x - 11$ or $3x - 2y - 11 = 0$.

We obtain the same equation by starting with

$$y + 1 = \frac{3}{2}(x - 3).$$

Slope Intercept Equation of a Line

If we know that the slope of a line is m and the line cuts the y -axis at $y = b$ then the equation of the line simplifies to

SLOPE + INTERCEPT EQUATION OF A LINE

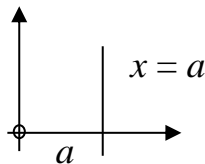
$$y = mx + b$$

This is because the line passes through $(0, b)$ and so its equation is $y - b = m(x - 0)$ which, on simplification, becomes $y = mx + b$.

Example 10: The equation of the line with slope 5 that cuts the y -axis at $y = -2$ is $y = 5x - 2$.

General Equation of a Line

All of the above methods rely in some way on the line having a slope. But a vertical line doesn't have a slope. We might say that it has *infinite* slope but this is not very useful since we can't use 'infinity' as if it was a number. If this line cuts the x -axis at $x = a$ then the x -coordinate of every point on the line will be a and so the equation of this vertical line will be $x = a$.



The general equation of a straight line is $px + qy + r = 0$ (with at least one of p, q non-zero)

If $p \neq 0$ and $q = 0$ this becomes $x = -\frac{r}{p}$ which represents a vertical line.

If $p \neq 0$ and $q \neq 0$ we can rewrite $px + qy + r = 0$ as

$$y = -\frac{p}{q}x - \frac{r}{q}.$$

If we put $m = -\frac{p}{q}$ and $b = -\frac{r}{q}$ this becomes the standard $y = mx + b$ form for the equation of a line.

Often the general equation is written as:

EQUATION OF A STRAIGHT LINE
 $ax + by + c = 0$

Clearly it doesn't matter which symbols we use. We used p, q, r at first so as not to confuse this b with the b in the $y = mx + b$ form.

Parallel Lines

If two lines are parallel they must have the same slope. So lines with slopes m_1 and m_2 are parallel if and only if $m_1 = m_2$.

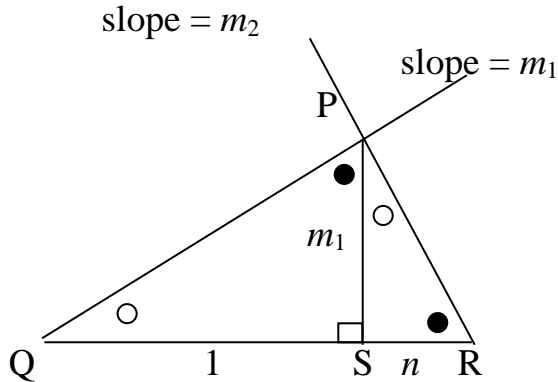
<p>PARALLEL LINES</p> <p>$m_1 = m_2$</p>
--

Perpendicular Lines

Suppose we have two perpendicular lines of slopes m_1 and m_2 respectively. The relationship between these two slopes is given by:

<p>PERPENDICULAR LINES</p> <p>$m_1 m_2 = -1$</p>
--

We can see this by using similar triangles.



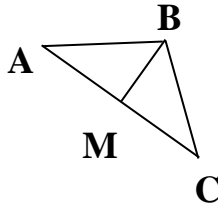
PS = m_1 , QS = 1 and RS = n . Since $\triangle PSQ$ is similar to $\triangle RSP$ (corresponding angles are equal) then $\frac{PS}{QS} = \frac{RS}{PS}$ (corresponding sides are proportional). So $\frac{m_1}{1} = \frac{n}{m_1}$ and hence $n = m_1^2$.

Now from triangle RSP, $m_2 = -\frac{m_1}{n} = -\frac{m_1}{m_1^2} = -\frac{1}{m_1}$ which gives $m_1 m_2 = -1$.

Example 11: The slope of any line parallel to $y = 4x + 7$ is 4. The slope of any line perpendicular to this line is $-1/4$.

Example 12: If A = (1, 7), B = (3, 7) and C = (5, 3) we can find the area of the triangle ABC using the “half base times perpendicular height” formula. Taking the base to be AC we need to calculate the distance between A and C. This is $\sqrt{16 + 16} = \sqrt{32}$.

But what is the perpendicular height? It’s BM where M is the foot of the perpendicular from B to the base AC.



The slope of AC is $\frac{3 - 7}{5 - 1} = -1$. So the equation of AC is

$y - 7 = -(x - 1)$, ie $y = -x + 8$.

Using the m_1m_2 relationship, we find that the slope of BM is 1. So the equation of M is:

$$y - 7 = x - 3, \text{ ie } y = x + 4.$$

Solving $y = -x + 8$ and $y = x + 4$ simultaneously we get $-x + 8 = x + 4$, so $2x = 4$ and so $x = 2$.

This gives $y = 6$, so M is (2, 6). Hence the perpendicular height is the length of $BM = \sqrt{1 + 1} = \sqrt{2}$.

The area is therefore $\frac{1}{2}\sqrt{32} \sqrt{2} = 4$.

[There are other methods for working out the area of a triangle that require less calculation but we did it this way to illustrate these techniques.]

EXERCISES FOR CHAPTER 2

Exercise 1:

Plot the following points: A(1, 3), B(-2, 4), C(5, -3), D(-2, -1).

Exercise 2: In which quadrants do the following points lie?

(i) (3, 2); (ii) (-3, -2); (iii) (2, -5); (iv) (-1, 7).

Exercise 3:

(a) Draw up a table of values for $y = x^2 + x + 1$, using the values $x = -3, -2, -1, 0, 1, 2, 3$.

(b) Draw the graph of this function.

(c) Use the graph to estimate the slope at $x = 1$.

(d) Use the Counting Squares Method to estimate the area under the graph from $x = 0$ to $x = 2$.

Exercise 4:

(a) Draw the graph of $y = \frac{2(1-x)}{1+x}$, from $x = -5$ to $x = 4$.

WARNING: It comes in two separate pieces.

(b) Use the graph to estimate the slope of the normal at $x = 1$.

(c) Find the equation of this normal. Draw it on your graph.

(d) It can be shown that the shortest distance between two curves occurs when the line joining the points is a normal

(perpendicular) to both curves. Use this fact to work out the shortest distance between the two halves of this curve.

Exercise 5: If $A = (1, 3)$ and $B = (4, 7)$ find:

- (i) the midpoint of AB ;
- (ii) the slope of AB ;
- (iii) the length of AB ;
- (iv) the equation of AB .

Exercise 6: A certain curve has slope 2 at the point $(1, 3)$. Find the equation of the tangent there.

Exercise 7: Find the equation of the line that cuts the x -axis at $x = 3$ and the y -axis at $y = 2$.

Exercise 8:

- (a) Find the slope of the line $5y - 2x + 7 = 0$.
- (b) Find the slope of any line that's perpendicular to the above line.

Exercise 9: Let A, B, C, D be the points in exercise 1.

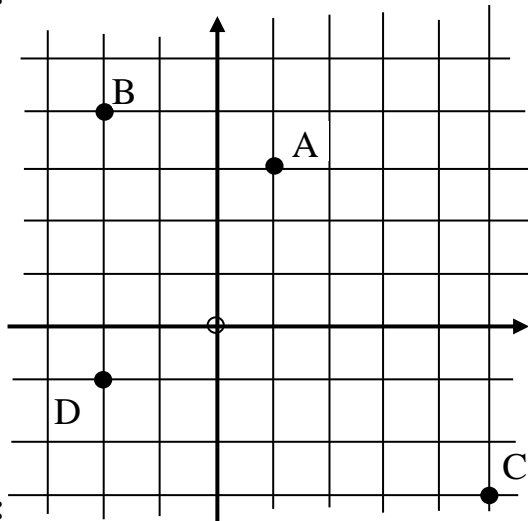
- (a) Find the length of CD .
- (b) Find the slope of AB .
- (c) Find the slope of any line that is perpendicular to AB .
- (d) Find the midpoint of AC .
- (e) Find the equation of AB .

Exercise 10: Let $A = (2, 5)$, $B = (3, 8)$, $C = (0, 4)$.

- (a) Find the slope of AC.
- (b) Find the equation of the line AC.
- (c) Find the equation of the line through B that is perpendicular to AC.
- (d) Find the point, M, where the two lines obtained above intersect.
(Solve the two equations simultaneously.)
- (e) Find the lengths of AC and BM.
- (f) Use the formula “half base times perpendicular height” to find the area of the triangle ABC.

SOLUTIONS FOR CHAPTER 2

Exercise 1:



Exercise 2:

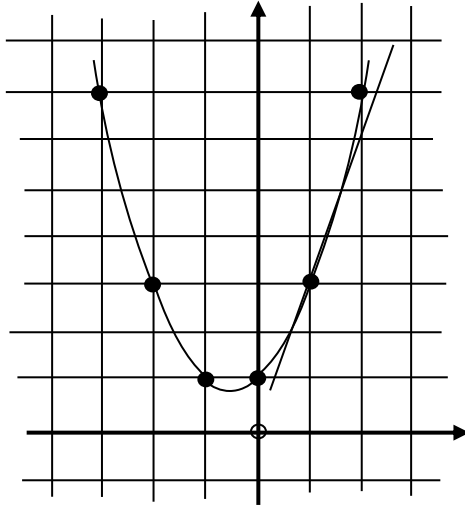
- (i) 1st
- (ii) 3rd
- (iii) 4th
- (iv) 2nd

Exercise 3:

(a)

x	-3	-2	-1	0	1	2
y	7	3	1	1	3	7

(b)



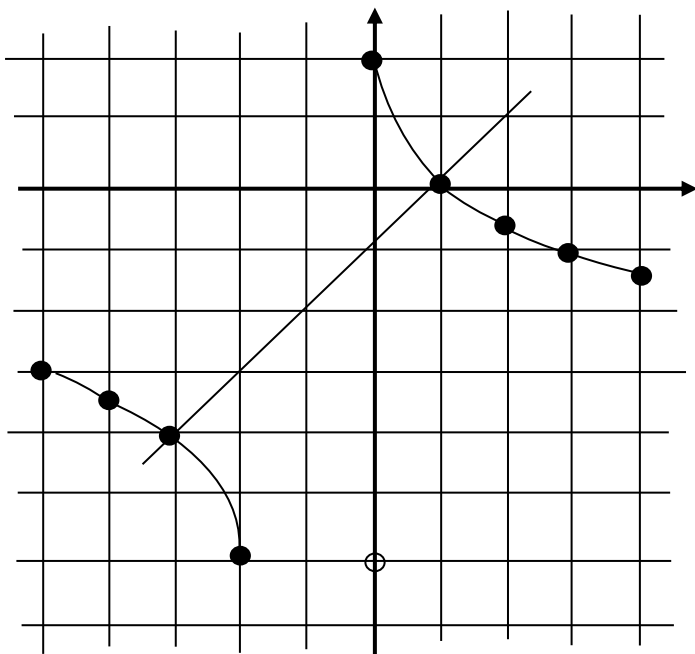
(c) The slope at $x = 1$ is approximately 3.

(d) The area under the graph from $x = 0$ to $x = 2$ is approximately 7.

Exercise 4:

(a)

x	-5	-4	-3	-2	-1	0	1	2	3	4
y	-3	-3.3	-4	-6	\times	2	0	-0.7	-1	-1.2



(b) The slope of the normal at $x = 1$ appears to be 1. (This is in fact the exact value.)

(c) Using this value, the equation of the normal at $x = 1$ is $y - 0 = (x - 1)$, which can be simplified to $y = x - 1$.

(d) The normal at $x = 1$ passes through $(-3, -4)$ and appears to be the normal at that point. (In fact calculus can tell us that it is.) So the shortest distance between the two halves of our graph is the distance between $(1, 0)$ and $(-3, -4)$.

This distance is $\sqrt{(-3 - 1)^2 + (-4 - 0)^2} = \sqrt{16 + 16}$
 $= \sqrt{32} = 4\sqrt{2}$.

Exercise 5:

(i) Midpoint = $\left(\frac{1+4}{2}, \frac{3+7}{2}\right) = (5/2, 5)$;

(ii) Slope = $\frac{7-3}{4-1} = \frac{4}{3}$;

(iii) Length = $\sqrt{(4-1)^2 + (7-3)^2} = 5$

(iv) $y - 3 = \frac{4}{3}(x - 1)$ which can be written as

$$4x - 3y + 5 = 0$$

Exercise 6: $y - 3 = 2(x - 1)$ which can be written as

$$2x - y + 1 = 0.$$

Exercise 7: The slope is $-\frac{2}{3}$. The line passes through

$(0, 2)$ so the equation is $y - 2 = -\frac{2}{3}(x - 0)$. This can be

simplified to $2x + 3y - 6 = 0$.

Exercise 8: Writing $5y - 2x + 7 = 0$ in the $y = mx + b$ form

we get $y = \frac{2}{5}x - \frac{7}{5}$. The slope is therefore $\frac{2}{5}$.

For perpendicular lines $m_1m_2 = -1$ so the slope of the

perpendicular is $-\frac{5}{2}$.

Exercise 9:

(a) $CD = \sqrt{(-2 - 5)^2 + (-1 + 3)^2} = \sqrt{49 + 4} = \sqrt{53}.$

(b) slope of AB = $\frac{4 - 3}{-2 - 1} = -\frac{1}{3}.$

(c) slope of a perpendicular to AB is 3.

(d) midpoint of AC is $\left(\frac{1 + 5}{2}, \frac{3 - 3}{2}\right) = (3, 0).$

(e) equation of AB is $y - 3 = -\frac{1}{3}(x - 1),$ which can be simplified to $x + 3y = 10.$ **Exercise 10:**

(a) Slope = $\frac{5 - 4}{2 - 0} = \frac{1}{2}$

(b) $y - 5 = \frac{1}{2}(x - 2)$ which can be written as

$$x - 2y + 8 = 0$$

(c) The slope of perpendicular is $-2,$ so the equation is $y - 8 = -2(x - 3)$ which can be written as $y = -2x + 14.$ (d) Solving $\left. \begin{array}{l} x - 2y + 8 = 0 \\ y = -2x + 14 \end{array} \right\}$ simultaneously we get $x = 4.$ Substituting back into the first equation we get $y = 6.$ So M is point (4,6).

(e) $AC = \sqrt{(0 - 2)^2 + (4 - 5)^2} = \sqrt{5}.$

$BM = \sqrt{(4 - 3)^2 + (6 - 8)^2} = \sqrt{5}.$

(f) The area of the triangle ABC = $\frac{1}{2} \cdot \sqrt{5} \cdot \sqrt{5} = \frac{5}{2}.$